RADIAL LIMIT SETS OF GRAPH DIRECTED MARKOV SYSTEMS ASSOCIATED TO FREE GROUPS

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ABSTRACT. For a conformal graph directed Markov system Φ associated to the free group $\mathbb{F}_d$ of rank $d \geq 2$, we investigate the radial limit set $L_r(N,\Phi)$ of non-trivial normal subgroups N of $\mathbb{F}_d$. Regarding the Hausdorff dimension of these sets, we show that various results for Kleinian groups have natural generalisations. We prove that if Φ is weakly symmetric, then the Hausdorff dimensions of $L_r(N,\Phi)$ and $L_r(\mathbb{F}_d,\Phi)$ coincide if and only if $\mathbb{F}_d/N$ is amenable. The corresponding result for Kleinian groups is due to Brooks and Bishop and Jones. Furthermore, we prove that $\dim_H L_r(N,\Phi) \geq \dim_H L_r(\mathbb{F}_d,\Phi)/2$ provided that Φ is weakly symmetric, and that strict inequality holds if Φ is symmetric, which extends work of Falk and Stratmann and by Roblin. Moreover, we prove that if the analogue of the Poincaré series of N is of divergence type, then $\dim_H L_r(N,\Phi) = \dim_H L_r(\mathbb{F}_d,\Phi)$ provided that Φ is weakly symmetric, which is similar to a result of Matsuzaki and Yabuki.

1. INTRODUCTION AND STATEMENT OF RESULTS

For a Kleinian group Γ acting on the Poincaré disc model $\mathbb{D} := \left\{z \in \mathbb{R}^{n+1} : \|z\| < 1\right\}$ of hyperbolic $(n+1)$-space, $n \in \mathbb{N}$, the Poincaré series is, for each $s \in \mathbb{R}$, given by $P(\Gamma,s) := \sum_{\gamma\in\Gamma} e^{-sd(0,\gamma(0))}$, where d denotes the hyperbolic metric on $\mathbb{D}$. The exponent of convergence of Γ is given by $\delta(\Gamma) := \inf\{s : P(\Gamma,s) < \infty\}$. It is well-known by a theorem of Bishop and Jones ([BJ97]) that the exponent of convergence of a non-elementary Kleinian group Γ is equal to the Hausdorff dimension of the (uniformly) radial limit set of Γ, that is

$$\delta(\Gamma) = \dim_H(L_{\mathrm{ur}}(\Gamma)) = \dim_H(L_{\mathrm{r}}(\Gamma)). \tag{1.1}$$

For references on limit sets of Kleinian groups and the associated hyperbolic manifolds, we refer the reader to [Bea95, Mas88, Nic89, MT98, Str06].

It is an interesting task to study how the exponent of convergence varies if one passes from a non-elementary Kleinian group G to a normal subgroup N of G, which gives rise to the normal covering $\mathbb{D}/N$ of $\mathbb{D}/G$. In [Bro85] it is shown that if G is convex cocompact and $\delta(G) > n/2$, then

$$\delta(G) = \delta(N) \text{ if and only if } G/N \text{ is amenable.} \tag{1.2}$$

A recent result in [Sta11] shows that the amenability dichotomy in (1.2) holds for all essentially free Kleinian groups G with arbitrary exponent of convergence. A complementary result is due to Falk and Stratmann [FS04, Theorem 2] which states that for each non-trivial normal subgroup N of a non-elementary Kleinian group G, we have that

$$\delta(N) \geq \frac{\delta(G)}{2}. \tag{1.3}$$

2000 *Mathematics Subject Classification.* Primary 37C45, 30F40 ; Secondary 37C85, 43A07.
Key words and phrases. Kleinian groups, exponent of convergence, normal subgroups, amenability, conformal graph directed Markov systems.
The author was supported by the research fellowship JA 2145/1-1 of the German Research Foundation (DFG).

Roblin [Rob05] proved that strict inequality in (1.3) holds if the Kleinian group G is of divergence type, that is $P(G,\delta(G))=\infty$. Another proof of this result was independently obtained by Bonfert-Taylor, Matsuzaki and Taylor [BTMT12] for G convex cocompact. A related result by Matsuzaki and Yabuki in [MY09] states that

$$\text{if } P(N,\delta(N))=\infty \text{ then } \delta(G)=\delta(N). \tag{1.4}$$

In [Jae12b], the author used this result of Matsuzaki and Yabuki to give a short new proof of the strict inequality in (1.3) if G is of divergence type.

The aim of this paper is to investigate a large class of conformal fractals associated to normal subgroups of free groups, and to prove properties similar to those stated in (1.1), (1.2), (1.3) and (1.4). In [Jae11a], the author has introduced the *radial limit set* and the *uniformly radial limit set* of a normal subgroup N of the free group $\mathbb{F}_d$ of rank $d\geq 2$ with respect to a *conformal graph directed Markov systems* Φ *(GDMS) associated to* $\mathbb{F}_d$. These sets are denoted by $L_{\mathrm{r}}(N,\Phi)$ and $L_{\mathrm{ur}}(N,\Phi)$ respectively. The *Poincaré series of* N *with respect to* Φ, and the *exponent of convergence of* N *with respect to* Φ are given by

$$P(N,s,\Phi):=\sum_{h\in N}(c_\Phi(h))^s,\ s\in\mathbb{R},\quad \delta(N,\Phi):=\inf\{s\geq 0: P(N,s,\Phi)<\infty\},$$

where $c_\Phi:\mathbb{F}_d\to\mathbb{R}$ denotes the *geometric weight function of* Φ. (We refer to Definition 3.8 for precise definitions.) In Proposition 3.12, we will show the analogue of (1.1), that is, for a conformal GDMS Φ associated to $\mathbb{F}_d$ with $d\geq 2$, and for a non-trivial normal subgroup N of $\mathbb{F}_d$, we have that

$$\dim_H(L_{\mathrm{r}}(N,\Phi))=\dim_H(L_{\mathrm{ur}}(N,\Phi))=\delta(N,\Phi).$$

In order to state our main results, a conformal GDMS Φ associated to $\mathbb{F}_d$ is defined to be *weakly symmetric* if there exists a sequence $(D_n)_{n\in\mathbb{N}}$ with $D_n\geq 1$ and $\lim_{n\to\infty}D_n^{1/n}=1$ such that for each $n\in\mathbb{N}$ and for all $g\in\mathbb{F}_d$ of word length n, we have that

$$D_n^{-1}\leq\frac{c_\Phi(g)}{c_\Phi(g^{-1})}\leq D_n,$$

and we say that Φ is symmetric if the sequence $(D_n)_{n\in\mathbb{N}}$ can be chosen to be bounded. These notions of symmetry, which naturally arise from the geometry of Kleinian groups, have been considered by Stadlbauer ([Sta11]) and by the author ([Jae11b, Jae11a]).

Theorem 1.1. *Let* Φ *denote a conformal graph directed Markov system associated to* $\mathbb{F}_d$ *with* $d\geq 2$. *Then, for each non-trivial normal subgroup* N of $\mathbb{F}_d$, the following holds.

(1) *If* $\delta(N,\Phi)=\delta(\mathbb{F}_d,\Phi)$, *then* $\mathbb{F}_d/N$ *is amenable.*
(2) *If* Φ *is weakly symmetric and* $\mathbb{F}_d/N$ *is amenable, then* $\delta(N,\Phi)=\delta(\mathbb{F}_d,\Phi)$.
(3) *If* Φ *is weakly symmetric, then* $\delta(N,\Phi)\geq\delta(\mathbb{F}_d,\Phi)/2$. If Φ *is symmetric, then* $\delta(N,\Phi)>\delta(\mathbb{F}_d,\Phi)/2$.
(4) *If* Φ *is weakly symmetric and* $P(N,\delta(N),\Phi)=\infty$, *then* $\delta(N,\Phi)=\delta(\mathbb{F}_d,\Phi)$.

As a corollary we derive that for finitely generated normal subgroups, the Poincaré series diverges, whereas, for normal subgroups N such that $\mathbb{F}_d/N$ is non-amenable, the Poincaré series converges.

Corollary 1.2. *In the situation of Theorem 1.1, suppose that* Φ *is weakly symmetric. If* N *is finitely generated, then* $P(N,\delta(N,\Phi),\Phi)=\infty$, $\delta(N,\Phi)=\delta(\mathbb{F}_d,\Phi)$ *and* $\mathbb{F}_d/N$ *is amenable. On the other hand, if* $\mathbb{F}_d/N$ *is non-amenable, then* N *is infinitely generated and* $P(N,\delta(N,\Phi),\Phi)<\infty$.

Note that Theorem 1.1 (1) and (2) extend the amenability dichotomy in (1.2). The lower bounds given in Theorem 1.1 (3) are similar to (1.3). Here, we remark that, since $\mathbb{F}_d$ is finitely generated, we have that $P(\mathbb{F}_d, \delta(\mathbb{F}_d, \Phi), \Phi)$ diverges by Corollary 1.2, which corresponds to the assumption that the group G is of divergence type in order for (1.3) to hold. Finally, Theorem 1.1 (4) provides the analogue of (1.4).

For a symmetric, conformal GDMS Φ consisting of similarities, the results stated in Theorem 1.1 (1), (2) and (3) are contained in [Jae11a]. In the present paper, we have generalised the results to arbitrary conformal GDMSs. Furthermore, we have partially extended the results to weakly symmetric GDMSs. Moreover, an analogue of the result of Matsuzaki and Yabuki is stated in Theorem 1.1 (4), which is also new in the case of GDMSs consisting of similarities.

The proofs of Theorem 1.1 (1) and (2) make use of the thermodynamic formalism for group-extended Markov shifts developed by Stadlbauer in [Sta11] and by the author in [Jae11b, Jae11a, Jae12c]. The results stated in Theorem 1.1 (3) and (4) make use of recent results of the author on recurrence and pressure for group-extended Markov shifts ([Jae12c]). In Proposition 4.7 in Section 4, we also formulate versions of Theorem 1.1 (3) and (4) in the abstract setting of the thermodynamic formalism of Markov shifts.

The outline of this paper is as follows. In Section 2, we collect the necessary preliminaries on symbolic thermodynamic formalism of Markov shifts. In Section 3, we recall the definition of conformal GDMSs associated to free groups, and of radial limit sets of normal subgroups. In Section 4 we study group-extended Markov systems from which we deduce our main results in Section 5.

2. Thermodynamic Formalism for Markov shifts

We consider a *Markov shift* Σ given by

$$\Sigma := \left\{ \omega := (\omega_1, \omega_2, \dots) \in I^{\mathbb{N}} : \ a(\omega_i, \omega_{i+1}) = 1 \text{ for all } i \in \mathbb{N} \right\},$$

where I denotes a finite or countable infinite *alphabet* I, the matrix $A = (a(i,j)) \in \{0,1\}^{I \times I}$ is the *incidence matrix* and the *shift map* $\sigma : \Sigma \to \Sigma$ is given by $(\omega_1, \omega_2, \dots) \mapsto (\omega_2, \omega_3, \dots)$. The set of A-admissible words of length $n \in \mathbb{N}$ is given by

$$\Sigma^n := \{ \omega \in I^n : \ a(\omega_i, \omega_{i+1}) = 1 \text{ for all } i \in \{1, \dots, n-1\} \}.$$

The set of A-admissible words of arbitrary length is given by $\Sigma^* := \bigcup_{n \in \mathbb{N}} \Sigma^n$. We define the *word length function* $|\cdot| : \Sigma^* \cup \Sigma \to \mathbb{N} \cup \{\infty\}$, where for $\omega \in \Sigma^*$ we set $|\omega|$ to be the unique $n \in \mathbb{N}$ such that $\omega \in \Sigma^n$ and for $\omega \in \Sigma$ we set $|\omega| := \infty$. For each $\omega \in \Sigma^* \cup \Sigma$ and $n \in \mathbb{N}$ with $1 \leq n \leq |\omega|$, we define $\omega_{|n} := (\omega_1, \dots, \omega_n)$. For $\omega, \tau \in \Sigma$, we let $\omega \wedge \tau := \omega_{|l}$, where $l := \sup\left\{ n \in \mathbb{N} : \omega_{|n} = \tau_{|n} \right\}$. For $\omega \in \Sigma^n$, $n \in \mathbb{N}_0$, the *cylinder set* $[\omega]$ defined by ω is given by $[\omega] := \left\{ \tau \in \Sigma : \tau_{|n} = \omega \right\}$.

If Σ is the Markov shift with alphabet I whose incidence matrix consists entirely of 1s, then we have that $\Sigma = I^{\mathbb{N}}$ and $\Sigma^n = I^n$, for all $n \in \mathbb{N}$. Then we set $I^* := \Sigma^*$. For $\omega, \tau \in I^*$ we denote by $\omega\tau \in I^*$ the *concatenation* of ω and τ, which is defined by $\omega\tau := \left(\omega_1, \dots, \omega_{|\omega|}, \tau_1, \dots, \tau_{|\tau|}\right)$ for $\omega, \tau \in I^*$. Note that I^* forms a semigroup with respect to the concatenation operation. The semigroup I^* is the free semigroup over the set I and satisfies the following universal property. For each semigroup S and for every map $u : I \to S$, there exists a unique semigroup homomorphism $\widehat{u} : I^* \to S$ such that $\widehat{u}(i) = u(i)$, for all $i \in I$ (see [Ber98, Section 3.10]).

We need the following mixing properties for a Markov shift Σ.

- Σ is *irreducible* if
$$\forall i,j\in I\,\exists\omega\in\Sigma^{*}:\,i\omega j\in\Sigma^{*}.$$
- Σ is *mixing* if
$$\forall i,j\in I\,\exists n_{0}\in\mathbb{N}\,\forall n\geq n_{0}\,\exists\omega\in\Sigma^{n}:\,i\omega j\in\Sigma^{*}.$$
- Σ is *finitely irreducible* if there exists a finite set $\Lambda\subset\Sigma^{*}$ such that
$$\forall i,j\in I\,\exists\omega\in\Lambda:\,i\omega j\in\Sigma^{*}.$$
- Σ is *finitely primitive* if there exists $l\in\mathbb{N}$ and a finite set $\Lambda\subset\Sigma^{l}$ such that
$$\forall i,j\in I\,\exists\omega\in\Lambda:\,i\omega j\in\Sigma^{*}.$$

Remark. Note that Σ is finitely primitive if and only if Σ is mixing and if Σ satisfies the big images and preimages (BIP) property (see [Sar03]).

We equip $I^{\mathbb{N}}$ with the product topology of the discrete topologies on I. The Markov shift $\Sigma\subset I^{\mathbb{N}}$ is equipped with the subspace topology. This topology on Σ is the weakest topology such that for each $j\in\mathbb{N}$ the canonical *projection on the j-th coordinate* $p_{j}:\Sigma\to I$ is continuous. A countable basis of the topology on Σ is given by the cylinder sets $\{[\omega]:\omega\in\Sigma^{*}\}$. We will make use of the following metrics which each generate the product topology on Σ. For $\alpha>0$ fixed, we define the metric d_{α} on Σ given by
$$d_{\alpha}(\omega,\tau):=\mathrm{e}^{-\alpha|\omega\wedge\tau|},\text{ for all }\omega,\tau\in\Sigma.$$

For a function $f:\Sigma\to\mathbb{R}$ and $n\in\mathbb{N}_{0}$ we use the notation $S_{n}f:\Sigma\to\mathbb{R}$ to denote the *ergodic sum* of f with respect to the left shift σ, in other words, $S_{n}f:=\sum_{i=0}^{n-1}f\circ\sigma^{i}$.

Definition 2.1. We say that a function $f:\Sigma\to\mathbb{R}$ is *α-Hölder continuous*, for some $\alpha>0$, if
$$V_{\alpha}(f):=\sup_{n\geq1}\{V_{\alpha,n}(f)\}<\infty,$$
where for each $n\in\mathbb{N}$ we let
$$V_{\alpha,n}(f):=\sup\left\{\mathrm{e}^{-\alpha}\frac{|f(\omega)-f(\tau)|}{d_{\alpha}(\omega,\tau)}:\omega,\tau\in\Sigma,|\omega\wedge\tau|\geq n\right\}.$$
The function f is *Hölder continuous* if there exists $\alpha>0$ such that f is α-Hölder continuous.

The following fact is taken from [MU03, Lemma 2.3.1].

Fact 2.2. *If $f:\Sigma\to\mathbb{R}$ is Hölder continuous, then there exists a constant $C_{f}>0$ such that*
$$\left|S_{|\omega|}f(\tau)-S_{|\omega|}f\left(\tau'\right)\right|\leq C_{f},\textit{ for all }\omega\in\Sigma^{*}\textit{ and }\tau,\tau'\in[\omega].$$

We need the following notion of pressure introduced in [JKL10, Definition 1.1].

Definition 2.3 (Induced topological pressure)**.** For $\varphi,\psi:\Sigma\to\mathbb{R}$ with $\psi\geq0$, and $\mathscr{C}\subset\Sigma^{*}$ we define for $\eta>0$ the *ψ-induced pressure of φ* (with respect to $\mathscr{C}$) by
$$\mathscr{P}_{\psi}(\varphi,\mathscr{C}):=\limsup_{T\to\infty}\frac{1}{T}\log\sum_{\substack{\omega\in\mathscr{C}\\ T-\eta<S_{\omega}\psi\leq T}}\exp S_{\omega}\varphi,$$
which takes values in $\overline{\mathbb{R}}:=\mathbb{R}\cup\{\pm\infty\}$. In here, we set $S_{\omega}\varphi:=\sup_{\tau\in[\omega]}S_{|\omega|}\varphi(\tau)$.

Remark. It was shown in [JKL10, Theorem 2.4] that the definition of $\mathscr{P}_\psi(\varphi,\mathscr{C})$ is in fact independent of the choice of $\eta>0$. For this reason we do not refer to $\eta>0$ in the definition of the induced pressure.

Notation. If ψ and/or $\mathscr{C}$ is left out in the notation of induced pressure, then we tacitly assume that $\psi=1$ and/or $\mathscr{C}=\Sigma^*$. That is, $\mathscr{P}(\varphi):=\mathscr{P}_1(\varphi,\Sigma^*)$.

The following fact is taken from [JKL10, Remark 2.11, Remark 2.7].

Fact 2.4. *Let Σ be finitely irreducible, $\mathscr{C}\subset\Sigma^*$ and let $\varphi,\psi:\Sigma\to\mathbb{R}$, $\psi\geq c>0$ for some $c>0$. Then we have*

$$\mathscr{P}_\psi(\varphi,\mathscr{C})=\inf\{s\in\mathbb{R}:\mathscr{P}(\varphi-s\psi,\mathscr{C})\leq 0\}=\inf\left\{s\in\mathbb{R}:\sum_{\omega\in\mathscr{C}}\mathrm{e}^{S_\omega(\varphi-\beta\psi)}<\infty\right\}.$$

If additionally $\operatorname{card}(I)<\infty$, *then $\mathscr{P}_\psi(\varphi,\mathscr{C})$ is the unique $s\in\mathbb{R}$ such that $\mathscr{P}(\varphi-s\psi,\mathscr{C})=0$.*

The following notion of a Gibbs measure is fundamental for the thermodynamic formalism (cf. [Rue69], [Bow75]).

Definition 2.5 (Gibbs measure)**.** Let $\varphi:\Sigma\to\mathbb{R}$ be a continuous. We say that a Borel probability measure μ *is a Gibbs measure for* φ if there exists a constant $C_\varphi>0$ such that

$$C_\varphi^{-1}\leq\frac{\mu[\omega]}{\mathrm{e}^{S_{|\omega|}\varphi(\tau)-|\omega|\mathscr{P}(\varphi)}}\leq C_\varphi,\ \text{for all }\omega\in\Sigma^*\text{ and }\tau\in[\omega]\,. \tag{2.1}$$

The following theorem is proved in [MU03, Section 2].

Theorem 2.6 (Existence of Gibbs measures)**.** *Let Σ be finitely irreducible and let $\varphi:\Sigma\to\mathbb{R}$ be Hölder continuous with $\mathscr{P}(\varphi)<\infty$. Then there exists a unique σ-invariant Gibbs measure μ_φ for φ.*

3. Graph Directed Markov Systems associated to Free Groups

In this section first recall the definition of a conformal graph directed Markov system (GDMS) introduced by Mauldin and Urbański [MU03]. We then recall the definition of GDMSs associated to free groups and their radial limit sets.

3.1. Preliminaries.

Definition 3.1. A *graph directed Markov system (GDMS)* $\Phi=\left(V,(X_v)_{v\in V},E,i,t,(\phi_e)_{e\in E},A\right)$ consists of a finite vertex set V, a family of nonempty compact metric spaces $(X_v)_{v\in V}$, a countable edge set E, maps $i,t:E\to V$, a family of injective contractions $\phi_e:X_{t(e)}\to X_{i(e)}$ with Lipschitz constants bounded by some $0<s<1$, and an edge incidence matrix $A\in\{0,1\}^{E\times E}$, which has the property that $a(e,f)=1$ implies $t(e)=i(f)$, for all $e,f\in E$. For a GDMS Φ there exists a canonical *coding map*

$$\pi_\Phi:\Sigma_\Phi\to\oplus_{v\in V}X_v\text{ such that }\bigcap_{n\in\mathbb{N}}\phi_{\omega_{|n}}\left(X_{t(\omega_n)}\right)=\{\pi_\Phi(\omega)\}\,,$$

where $\oplus_{v\in V}X_v$ denotes the disjoint union of the sets X_v and Σ_Φ denotes the Markov shift with alphabet set E and incidence matrix A. The set

$$J(\Phi):=\pi_\Phi(\Sigma_\Phi)$$

refers to the *limit set of* Φ. We also set

$$J^*(\Phi) := \bigcup_{F\subset E,\,\mathrm{card}(F)<\infty} \pi_\Phi\left(\Sigma_\Phi \cap F^{\mathbb{N}}\right).$$

The following was introduced in [MU03, Section 4].

Definition 3.2 (Conformal GDMS)**.** The GDMS $\Phi = \left(V, (X_v)_{v\in V}, E, i, t, (\phi_e)_{e\in E}, A\right)$ is called *conformal* if the following conditions are satisfied.

(a) For $v \in V$, the *phase space* X_v is a compact connected subset of a Euclidean space $\left(\mathbb{R}^D, \|\cdot\|\right)$, for some $D \geq 1$, such that X_v is equal to the closure of its interior, that is $X_v = \overline{\mathrm{Int}(X_v)}$.

(b) (*Open set condition* (OSC)) For all $a, b \in E$ with $a \neq b$, we have that

$$\phi_a\left(\mathrm{Int}(X_{t(a)})\right) \cap \phi_b\left(\mathrm{Int}(X_{t(b)})\right) = \varnothing.$$

(c) For each vertex $v \in V$ there exists an open connected set $W_v \supset X_v$ such that the map ϕ_e extends to a C^1 conformal diffeomorphism of W_v into $W_{i(e)}$, for every $e \in E$ with $t(e) = v$.

(d) (*Cone property*) There exist $l > 0$ and $0 < \gamma < \pi/2$ such that for each $x \in X \subset \mathbb{R}^D$ there exists an open cone $\mathrm{Con}(x, \gamma, l) \subset \mathrm{Int}(X)$ with vertex x, central angle of measure γ and altitude l.

(e) There are two constants $L \geq 1$ and $\alpha > 0$ such that for each $e \in E$ and every $x, y \in X_{t(e)}$ we have

$$\left|\left|\phi_e'(y)\right| - \left|\phi_e'(x)\right|\right| \leq L \inf_{u\in W_{t(e)}} \left|\phi_e'(u)\right| \|y - x\|^\alpha.$$

Next lemma follows from (e) and is taken from [MU03, Lemma 4.2.2].

Lemma 3.3. *If Φ is a conformal GDMS, then for all $\omega \in \Sigma_\Phi^*$ and for all $x, y \in W_{t(\omega)}$, we have*

$$\left|\log\left|\phi_\omega'(y)\right| - \log\left|\phi_\omega'(x)\right|\right| \leq \frac{L}{1-s}\|x - y\|^\alpha. \tag{3.1}$$

Definition 3.4. For a GDMS Φ satisfying (a) and (c) of Definition 3.2, the *associated geometric potential* $\zeta_\Phi : \Sigma_\Phi \to \mathbb{R}^-$ is given by

$$\zeta_\Phi(\omega) := \log\left|\phi_{\omega_1}'\left(\pi_\Phi\left(\sigma(\omega)\right)\right)\right|, \text{ for all } \omega \in \Sigma_\Phi.$$

The following fact follows from [MU03, Proposition 4.2.7, Lemma 3.1.3] and Lemma 3.3.

Fact 3.5. *For a GDMS Φ which satisfies (a) and (c) of Definition 3.2, and for which (3.1) of Lemma 3.3 holds, the associated geometric potential ζ_Φ is Hölder continuous. In particular, this holds for a conformal GDMS.*

The following result is taken from [MU03, Theorem 4.2.13], where finitely primitivity can be replaced by finitely irreducibility (see also [RU08, Theorem 3.7]). We added the last equality in Theorem 3.6 which follows from Fact 2.4 since $-\zeta_\Phi$ is bounded away from zero.

Theorem 3.6 (Generalised Bowen's formula)**.** *Let Φ be a conformal GDMS with a finitely irreducible incidence matrix A and let $\zeta_\Phi : \Sigma_\Phi \to \mathbb{R}^-$ denote the associated geometric potential. We then have that*

$$\dim_H(J(\Phi)) = \dim_H(J^*(\Phi)) = \inf\{s \in \mathbb{R} : \mathscr{P}(s\zeta_\Phi) \leq 0\} = \mathscr{P}_{-\zeta_\Phi}\left(0, \Sigma_\Phi^*\right).$$

Remark 3.7. The generalised Bowen's formula also holds if the GDMS Φ satisfies (a)-(d) of Definition 3.2 and (3.1) of Lemma 3.3. In the case of $D \geq 2$, it follows from [MU03, Proposition 4.2.1]

that if Φ satisfies (a) and (c) of Definition 3.2, then Φ satisfies automatically (e) with $\alpha = 1$. In the remaining case $D = 1$, a closer inspection of the proof of [MU03, Theorem 4.2.13] and its preliminaries in [MU03, Section 4.2] shows that Definition 3.2 (e) is in fact only used to deduce (3.1) of Lemma 3.3, from which the bounded distortion property of Φ follows.

3.2. **Radial limit sets.** In the following we recall the definition of a GDMS Φ associated to the free group $\mathbb{F}_d = \langle g_1, \dots, g_d \rangle$ of rank $d \geq 2$, and the definition of the radial limit set of a normal subgroup N of $\mathbb{F}_d$ (see [Jae11a, Definition 2.11]).

Definition 3.8 (GDMS associated to free groups, geometric weight function, radial limit sets)**.** The GDMS $\Phi = \left(V, (X_v)_{v \in V}, E, i, t, (\phi_e)_{e \in E}, A\right)$ is *associated to* $\mathbb{F}_d = \langle g_1, \dots, g_d \rangle$, $d \geq 2$, if $V = \left\{g_1^{\pm 1}, \dots, g_d^{\pm 1}\right\}$, $E = \left\{(v, w) \in V^2 : v \neq w^{-1}\right\}$, $i, t : E \to V$ are given by $i(v, w) = v$ and $t(v, w) = w$ and the incidence matrix $A \in \{0,1\}^{E \times E}$ satisfies $a(e, f) = 1$ if and only if $t(e) = i(f)$, for all $e, f \in E$. The *geometric weight function* $c_\Phi : \mathbb{F}_d \to \mathbb{R}$ is for each $g \in \mathbb{F}_d$ given by

$$c_\Phi(g) := \sup\left\{\mathrm{e}^{S_\omega \zeta_\Phi} : \omega \in \Sigma_\Phi^* \text{ such that } i(\omega_1) \cdot \dots \cdot i\left(\omega_{|\omega|}\right) = g\right\}.$$

For a normal subgroup N of $\mathbb{F}_d$ and a GDMS Φ associated to $\mathbb{F}_d$, the *radial* and the *uniformly radial limit set of N with respect to* Φ are given by

$$L_{\mathrm{r}}(N, \Phi) := \pi_\Phi \left\{(v_i, w_i) \in \Sigma_\Phi : \exists h \in \mathbb{F}_d \text{ such that for infinitely many } n \in \mathbb{N}, v_1 \cdot \dots \cdot v_n \in hN\right\},$$

and

$$L_{\mathrm{ur}}(N, \Phi) := \pi_\Phi \left\{(v_i, w_i) \in \Sigma_\Phi : \exists H \subset \mathbb{F}_d \text{ finite such that for all } n \in \mathbb{N}, v_1 \cdot \dots \cdot v_n \in HN\right\}.$$

Before we proceed to analyse radial limit sets of normal subgroups of GDMSs associated to free groups, we think that some motivation for these definitions is in order. Conformal GDMSs can be used to describe the limit set of Kleinian groups of Schottky type $G = \langle g_1, \dots, g_d \rangle$ (see [MU03, Theorem 5.1.6]). These groups are free groups generated by hyperbolic transformations. The combinatorics of the associated conformal GDMS Φ_G captures the structure of the Cayley graph of the free group $\mathbb{F}_d$, and the symbolic representation of the limit set is given by the space of ends of this graph. We say that this type of GDMS is associated to the free group. Moreover, for each non-trivial normal subgroup N of G, it is shown in [Jae11a, Proposition 3.5] (see also [Jae11b, Corollary 6.2.9]) that

$$L_{\mathrm{r}}(N) = L_{\mathrm{r}}(N, \Phi_G) \text{ and } L_{\mathrm{ur}}(N) = L_{\mathrm{ur}}(N, \Phi_G),$$

which motivates the definition of the radial limit set of normal subgroups of GDMSs associated to free groups.

3.3. **The induced GDMS.** In order to investigate the radial limit set of a normal subgroup N of $\mathbb{F}_d$, we introduce an induced GDMS $\tilde{\Phi}$ whose edges consist of first return loops in the Cayley graph of $\mathbb{F}_d/N$.

Definition 3.9. Let $\Phi = \left(V, (X_v)_{v \in V}, E, i, t, (\phi_e)_{e \in E}, A\right)$ denote a conformal GDMS associated to $\mathbb{F}_d$ with $d \geq 2$, and let N denote a non-trivial normal subgroup of $\mathbb{F}_d$. The N-*induced* GDMS $\tilde{\Phi} := \left(V, (X_v)_{v \in V}, \tilde{E}, \tilde{i}, \tilde{t}, \left(\tilde{\phi}_\omega\right)_{\omega \in \tilde{E}}, \tilde{A}\right)$ is given by

$$\tilde{E} := \left\{\omega = (v_i, w_i) \in \Sigma_\Phi^* : v_1 \cdot \dots \cdot v_{|\omega|} \in N,\ v_1 \cdot \dots \cdot v_k \notin N \text{ for all } 1 \leq k < |\omega|\right\},$$

$\tilde{i}, \tilde{t} : \tilde{E} \to V$, $\tilde{i}(\omega) := i(\omega_1)$, $\tilde{t}(\omega) := t\left(\omega_{|\omega|}\right)$ for each $\omega \in \tilde{E}$, incidence matrix $\tilde{A} \in \{0,1\}^{\tilde{E} \times \tilde{E}}$ such that $\tilde{a}(\omega, \omega') = 1$ if and only if $a\left(\omega_{|\omega|}, \omega_1'\right) = 1$, and $\left(\tilde{\phi}_\omega\right)_{\omega \in \tilde{E}}$, $\tilde{\phi}_\omega := \phi_{\omega_1} \circ \dots \circ \phi_{\omega_{|\omega|}}$.

Notation 3.10. For the N-induced GDMS $\tilde{\Phi}$, there are canonical embeddings from $\Sigma^*_{\tilde{\Phi}}$ into Σ^*_{Φ}, and from $\Sigma_{\tilde{\Phi}}$ into Σ_{Φ}, which we will both indicate by omitting the tilde, that is $\tilde{\omega} \mapsto \omega$. It will always be clear which map is in use.

The proof of the following lemma is straightforward, and therefore left to the reader.

Lemma 3.11. *Let Φ denote a conformal GDMS associated to $\mathbb{F}_d$ with $d \geq 2$. Let N denote a non-trivial normal subgroup of $\mathbb{F}_d$, and let $\tilde{\Phi}$ denote the N-induced GDMS. We then have the following.*

(1) *The incidence matrix $\tilde{A}$ of $\tilde{\Phi}$ is finitely irreducible.*
(2) *There is a canonical bijection between $\Sigma^*_{\tilde{\Phi}}$ and $\mathscr{C}_N := \left\{\omega = (v_i, w_i) \in \Sigma^*_{\Phi} : v_1 \cdot \dots \cdot v_{|\omega|} \in N\right\}$.*
(3) *For the coding maps $\pi_{\tilde{\Phi}} : \Sigma_{\tilde{\Phi}} \to J\left(\tilde{\Phi}\right)$ and $\pi_{\Phi} : \Sigma_{\Phi} \to J(\Phi)$, we have $\pi_{\tilde{\Phi}}(\tilde{\omega}) = \pi_{\Phi}(\omega)$ for each $\tilde{\omega} \in \Sigma_{\tilde{\Phi}}$.*
(4) *The associated geometric potential $\zeta_{\tilde{\Phi}} : \Sigma_{\tilde{\Phi}} \to \mathbb{R}$ is Hölder continuous and for each $\tilde{\omega} \in \Sigma^*_{\tilde{\Phi}}$, we have $S_{\tilde{\omega}}\zeta_{\tilde{\Phi}} = S_{\omega}\zeta_{\Phi}$.*

The next proposition clarifies the relation between the Hausdorff dimension of the radial limit set and the exponent of convergence of a normal subgroup N of $\mathbb{F}_d$. We extend [Jae11b, Proposition 6.2.8] and [Jae11a, Proposition 1.3]. We also observe that the N-induced GDMS $\tilde{\Phi}$ is regular in the sense of [MU03, Section 4, page 78] if and only if the Poincaré series of N is of divergence type. Recall that $\tilde{\Phi}$ is regular if there exists $t \in \mathbb{R}$ such that $\mathscr{P}(t\zeta_{\tilde{\Phi}}) = 0$.

Proposition 3.12. *Let Φ denote a conformal GDMS associated to $\mathbb{F}_d$* with $d \geq 2$, *and let N denote a non-trivial normal subgroup of $\mathbb{F}_d$. Let $\tilde{\Phi}$ denote the N-induced GDMS. We then have that*

$$\dim_H(L_{\mathrm{ur}}(N,\Phi)) = \dim_H(L_{\mathrm{r}}(N,\Phi)) = \dim_H\left(J\left(\tilde{\Phi}\right)\right) = \delta(N,\Phi) = \mathscr{P}_{-\zeta_{\tilde{\Phi}}}\left(0, \Sigma^*_{\tilde{\Phi}}\right).$$

Moreover, we have that $P(N, \delta(N,\Phi), \Phi) = \infty$ if and only if $\mathscr{P}\left(\delta(N,\Phi)\zeta_{\tilde{\Phi}}, \Sigma^*_{\tilde{\Phi}}\right) = 0$.

Proof. Let us first relate the limit set of the N-induced GDMS $\tilde{\Phi}$ to the radial limit set of N with respect to Φ. Using Lemma 3.11 (3), it is straightforward to verify that

$$J^*\left(\tilde{\Phi}\right) \subset L_{\mathrm{ur}}(N,\Phi) \subset L_{\mathrm{r}}(N,\Phi) \subset J\left(\tilde{\Phi}\right) \cup \bigcup_{\eta \in \Sigma^*_{\Phi}, \tilde{\omega} \in \Sigma_{\tilde{\Phi}} : \eta\omega \in \Sigma_{\Phi}} \phi_{\eta}\left(\pi_{\tilde{\Phi}}(\tilde{\omega})\right). \tag{3.2}$$

Note that the right-hand side of (3.2) is a countable union of Lipschitz continuous images of $J\left(\tilde{\Phi}\right)$. Since Lipschitz continuous maps do not increase Hausdorff dimension and since Hausdorff dimension is stable under countable unions, we obtain that

$$\dim_H\left(J^*\left(\tilde{\Phi}\right)\right) \leq \dim_H(L_{\mathrm{ur}}(N,\Phi)) \leq \dim_H(L_r(N,\Phi)) \leq \dim_H\left(J\left(\tilde{\Phi}\right)\right).$$

The GDMS $\tilde{\Phi}$ satisfies the conditions (a)-(d) in Definition 3.2. Further, since $\left(\log\tilde{\phi}'_{\tilde{\omega}}\right)_{\tilde{\omega}\in\Sigma^*_{\tilde{\Phi}}}$ is a subfamily of $(\log\phi'_{\omega})_{\omega\in\Sigma^*_{\Phi}}$, it follows that $\tilde{\Phi}$ satisfies (3.1) of Lemma 3.7. Moreover, by Lemma 3.11 (1), the incidence matrix of $\tilde{\Phi}$ is finitely irreducible. Hence, by Remark 3.7, the generalised Bowen's formula in Theorem 3.6 applies, and we have that

$$\dim_H(L_{\mathrm{ur}}(N,\Phi)) = \dim_H(L_r(N,\Phi)) = \mathscr{P}_{-\zeta_{\tilde{\Phi}}}\left(0, \Sigma^*_{\tilde{\Phi}}\right). \tag{3.3}$$

It remains to show that $\mathscr{P}_{-\zeta_{\tilde{\Phi}}}\left(0,\Sigma_{\tilde{\Phi}}^{*}\right)=\delta\left(N,\Phi\right)$. By Lemma 3.11 (2) and (4) we conclude that $\mathscr{P}_{-\zeta_{\tilde{\Phi}}}\left(0,\Sigma_{\tilde{\Phi}}^{*}\right)=\mathscr{P}_{-\zeta_{\Phi}}\left(0,\mathscr{C}_{N}\right)$. Since Σ is finitely irreducible, we have by Fact 2.4 that

$$\mathscr{P}_{-\zeta_{\Phi}}\left(0,\mathscr{C}_{N}\right)=\inf\left\{s\in\mathbb{R}:\sum_{\omega\in\mathscr{C}_{N}}\mathrm{e}^{sS_{\omega}\zeta_{\Phi}}<\infty\right\}.$$

Finally, since the map from $\mathscr{C}_{N}$ onto N given by $\omega=\left(\left(v_{1},w_{1}\right),\left(v_{2},w_{2}\right),\ldots,\left(v_{n},w_{n}\right)\right)\mapsto v_{1}v_{2}\cdots v_{n}$, $n\in\mathbb{N}$, is $(2d-1)$-to-one, and since for all $\omega\in\mathscr{C}_{N}$ we have $S_{\omega}\zeta_{\Phi}=c_{\Phi}\left(v_{1}\ldots v_{n}\right)$, it follows that

$$\inf\left\{s\in\mathbb{R}:\sum_{\omega\in\mathscr{C}_{N}}\mathrm{e}^{sS_{\omega}\zeta_{\Phi}}<\infty\right\}=\inf\left\{s\in\mathbb{R}:\sum_{g\in N}\left(c_{\Phi}\left(g\right)\right)^{s}<\infty\right\}=\delta\left(N,\Phi\right),$$

which completes the proof of the first assertion.

We now turn our attention to the second assertion. First suppose that $P\left(N,\delta\left(N,\Phi\right),\Phi\right)=\infty$. Since the series diverges, we have that $\mathscr{P}\left(\delta\left(N,\Phi\right)\zeta_{\tilde{\Phi}},\Sigma_{\tilde{\Phi}}^{*}\right)\geq0$. On the other hand, as seen before, we have that $\mathscr{P}_{-\zeta_{\tilde{\Phi}}}\left(0,\Sigma_{\tilde{\Phi}}^{*}\right)=\delta\left(N,\Phi\right)$. Since $\Sigma_{\tilde{\Phi}}$ is finitely irreducible by Lemma 3.11 (1), we have that $\mathscr{P}_{-\zeta_{\tilde{\Phi}}}\left(0,\Sigma_{\tilde{\Phi}}^{*}\right)=\inf\left\{s\in\mathbb{R}:\mathscr{P}\left(s\zeta_{\tilde{\Phi}},\Sigma_{\tilde{\Phi}}^{*}\right)\leq0\right\}$ by Fact 2.4. Using again that $\Sigma_{\tilde{\Phi}}$ is finitely irreducible, it follows from [MU03, Theorem 2.1.5] that the map $s\mapsto\mathscr{P}\left(s\zeta_{\tilde{\Phi}},\Sigma_{\tilde{\Phi}}^{*}\right)\in\mathbb{R}\cup\{\infty\}$ is the monotone limit of a sequence of continuous functions. Consequently, the map $s\mapsto\mathscr{P}\left(s\zeta_{\tilde{\Phi}},\Sigma_{\tilde{\Phi}}^{*}\right)$ is lower-semicontinuous, which then implies that $\mathscr{P}\left(\delta\left(N,\Phi\right)\zeta_{\tilde{\Phi}},\Sigma_{\tilde{\Phi}}^{*}\right)\leq0$. We have thus shown that $\mathscr{P}\left(\delta\left(N,\Phi\right)\zeta_{\tilde{\Phi}},\Sigma_{\tilde{\Phi}}^{*}\right)=0$.

Now suppose that $\mathscr{P}\left(\delta\left(N,\Phi\right)\zeta_{\tilde{\Phi}},\Sigma_{\tilde{\Phi}}^{*}\right)=0$. Since $\Sigma_{\tilde{\Phi}}$ is finitely irreducible, it follows from Theorem 2.6 that there exists a Gibbs measure μ on $\Sigma_{\tilde{\Phi}}^{*}$ for the potential $\delta\left(N,\Phi\right)\zeta_{\tilde{\Phi}}$. Using our assumption that $\mathscr{P}\left(\delta\left(N,\Phi\right)\zeta_{\tilde{\Phi}},\Sigma_{\tilde{\Phi}}^{*}\right)=0$, we conclude that there exists a constant $C>0$ such that

$$P\left(N,\delta\left(N,\Phi\right),\Phi\right)=\sum_{g\in N}\left(c_{\Phi}\left(g\right)\right)^{\delta(N,\Phi)}\geq C\sum_{n\in\mathbb{N}}\sum_{\tilde{\omega}\in\Sigma_{\tilde{\Phi}}^{n}}\mu\left[\tilde{\omega}\right]=\infty,$$

which completes the proof. □

4. Group Extended Markov systems

In this section $\Psi:I^{*}\to G$ is a semigroup homomorphism, where I^{*} denotes the free semigroup generated by I with respect to the concatenation of words, and G denotes a discrete group.

We consider the skew product dynamics $\sigma\rtimes\Psi:\Sigma\times G\to\Sigma\times G$, which is given by

$$\left(\sigma\rtimes\Psi\right)\left(\omega,g\right):=\left(\sigma\left(\omega\right),g\Psi\left(\omega_{1}\right)\right),\text{ for all }\left(\omega,g\right)\in\Sigma\times G.$$

We say that $(\Sigma\times G,\sigma\rtimes\Psi)$ is a *group-extended Markov system* (see also [Jae11a, Section 4]). Note that $(\Sigma\times G,\sigma\rtimes\Psi)$ is conjugated to the Markov shift with state space

$$\left\{\left(\left(\omega_{j},g_{j}\right)\right)_{j\in\mathbb{N}}\in\left(I\times G\right)^{\mathbb{N}}:\left(\omega_{j}\right)_{j\in\mathbb{N}}\in\Sigma,\forall i\in\mathbb{N}g_{i}\Psi\left(\omega_{i}\right)=g_{i+1}\right\}.$$

We denote the projections to the first and the second factor of $\Sigma\times G$ by $\pi_{1}:\Sigma\times G\to\Sigma$ and $\pi_{2}:\Sigma\times G\to G$, respectively.

A straightforward generalisation of the proof of [Jae11b, Remark 5.1.6] gives the following relation between $\mathscr{P}\left(\varphi,\Psi^{-1}\left\{\mathrm{id}\right\}\cap\Sigma^{*}\right)$ and the Gurevič pressure $\mathscr{P}\left(\varphi\circ\pi_{1},\sigma\rtimes\Psi\right)$ of $\varphi\circ\pi_{1}$ with respect to $(\Sigma\times G,\sigma\rtimes\Psi)$ (see [Sar99] for the definition of the Gurevič pressure).

Fact 4.1. *Let Σ be finitely primitive and $\varphi : \Sigma \to \mathbb{R}$ Hölder continuous. For an irreducible group-extended Markov system $(\Sigma \times G, \sigma \rtimes \Psi)$ we have that*

$$\mathscr{P}\left(\varphi, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right) = \mathscr{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi).$$

4.1. **Amenability of the group** G. Let us first recall the definition of the important property of groups which was introduced by von Neumann [Neu29] under the German name *messbar*. By Day [Day49], groups with this property were renamed amenable groups.

Definition 4.2. A discrete group G is *amenable* if there exists a finitely additive probability measure ν on the set of all subsets of G which is invariant under left multiplication by elements of G, that is we have that $\nu(A) = \nu(g(A))$ for all $g \in G$ and $A \subset G$.

We need the following notion of symmetry.

Definition 4.3. Let $(\Sigma \times G, \sigma \rtimes \Psi)$ denote a group-extended Markov system. Let $\varphi : \Sigma \to \mathbb{R}$ be Hölder continuous. We say that φ is *asymptotically symmetric with respect to* Ψ ([Jae11b, Definition 5.2.21]) if there exist sequences $(c_m)_{m\in\mathbb{N}} \in (\mathbb{R}^+)^{\mathbb{N}}$ and $(N_m)_{m\in\mathbb{N}} \in \mathbb{N}_0^{\mathbb{N}}$ with the property that $\lim_m (c_m)^{1/m} = 1$, $\lim_m m^{-1} N_m = 0$ and such that for each $g \in G$ and for all $n \in \mathbb{N}$ we have

$$\sum_{\omega \in \Sigma^n \cap \Psi^{-1}\{g\}} \mathrm{e}^{S_\omega \varphi} \le c_n \sum_{\omega \in \Sigma^* \cap \Psi^{-1}\left\{g^{-1}\right\} : n \le |\omega| \le n + N_n} \mathrm{e}^{S_\omega \varphi}.$$

If (c_m) can be chosen to be bounded, then we say that φ is *symmetric* with respect to Ψ.

A proof of the following theorem can be found in [Jae11b, Theorem 5.3.11], see also [Jae11a, Corollary 4.22 and Remark 4.23]. The case of an infinite alphabet has been considered in [Sta11, Theorem 4.1] and [Jae12a, Corollary 1.4 (2)].

Theorem 4.4. *Let* $\mathrm{card}(I) < \infty$ *and let* $(\Sigma \times G, \sigma \rtimes \Psi)$ *be an irreducible group-extended Markov system. Suppose that* $\varphi : \Sigma \to \mathbb{R}$ *is Hölder continuous and that* φ *is asymptotically symmetric with respect to* Ψ. *If* G *is amenable then we have* $\mathscr{P}\left(\varphi, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right) = \mathscr{P}(\varphi)$.

The next theorem provides a converse of Theorem 4.4 and is due to Stadlbauer [Sta11, Theorem 5.4].

Theorem 4.5. *Let* Σ *be finitely primitive and let* $(\Sigma \times G, \sigma \rtimes \Psi)$ *be an irreducible group-extended Markov system. Suppose that* $\varphi : \Sigma \to \mathbb{R}$ *is Hölder continuous with* $\mathscr{P}\left(\varphi, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right) < \infty$. *If* $\mathscr{P}\left(\varphi, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right) = \mathscr{P}(\varphi)$, *then* G *is amenable.*

For locally constant potentials on a finite state Markov shift Σ, the previous theorem was proved in [Jae11a, Theorem 1.1] using different methods. For $\Sigma = I^{\mathbb{N}}$ and φ depending only on the first coordinate, the results in Theorem 4.4 and Theorem 4.5 were proved by Kesten in [Kes59b, Kes59a].

Problem. For a non-amenable group G, it would be interesting to know if $\mathscr{P}\left(\varphi, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right)$ can be related to the spectrum of the Perron-Frobenius operator associated to the skew product dynamics $\sigma \rtimes \Psi$ with respect to the potential $\varphi \circ \pi_1 : \Sigma \times G \to \mathbb{R}$. For potentials φ depending only on a finite number of symbols, it was shown in [Jae11a] that $\mathscr{P}\left(\varphi, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right)$ can be related to the logarithm of the spectral radius of this Perron-Frobenius operator acting on a certain L^2-space (see [Jae11a, Theorem 4.24]).

4.2. **Recurrence and lower bounds for pressure.** Let Σ be finitely primitive and let $\varphi : \Sigma \to \mathbb{R}$ be Hölder continuous. Let $(\Sigma \times G, \sigma \rtimes \Psi)$ denote an irreducible group-extended Markov system. Recall that $\varphi \circ \pi_1 : \Sigma \times G \to \mathbb{R}$ is *recurrent* (with respect to $\sigma \rtimes \Psi$) if $\mathscr{P}\left(\varphi, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right) < \infty$ and

$$\sum_{n \in \mathbb{N}} \mathrm{e}^{-n\mathscr{P}\left(\varphi, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right)} \sum_{\omega \in \Sigma^n \cap \Psi^{-1}\{\mathrm{id}\}} \mathrm{e}^{S_\omega \varphi} = \infty.$$

Since $\mathscr{P}\left(\varphi, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right)$ coincides with the Gurevič pressure of $\varphi \circ \pi_1$ with respect to $\sigma \rtimes \Psi$, this definition of recurrence in fact coincides with Sarig's definition of recurrent potentials given in [Sar01, Definition 1].

In order to give lower bounds on $\mathscr{P}\left(\varphi, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right)$, we need the following theorem from [Jae12c, Theorem 1.1, Remark 1.2]. We have replaced the assumption $\mathscr{P}(\varphi) < \infty$ in [Jae12c, Theorem 1.1] by the weaker assumption $\mathscr{P}\left(\varphi, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right) < \infty$. Looking at the proof of [Jae12c, Theorem 1.1], one immediately verifies that we have in fact only used this weaker assumption.

Theorem 4.6. *Let Σ be finitely primitive and let $(\Sigma \times G, \sigma \rtimes \Psi)$ be an irreducible group-extended Markov system. Suppose that $\varphi : \Sigma \to \mathbb{R}$ is Hölder continuous with $\mathscr{P}\left(\varphi, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right) < \infty$. If φ is asymptotically symmetric with respect to Ψ and if $\varphi \circ \pi_1$ is recurrent, then $\mathscr{P}\left(\varphi, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right) = \mathscr{P}(\varphi)$.*

The next result gives a lower bound on $\mathscr{P}\left(\varphi, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right)$. A similar result to the first assertion is given in the author's thesis [Jae11b, Theorem 5.3.11]. The second assertion is inspired by [Jae11a, Lemma 5.1], where a locally constant potential φ is considered, and makes use of Theorem 4.6 above.

Proposition 4.7. *Let Σ be finitely primitive and let $(\Sigma \times G, \sigma \rtimes \Psi)$ be an irreducible group-extended Markov system. For each Hölder continuous potential $\varphi : \Sigma \to \mathbb{R}$ the following holds.*

(1) *If φ is asymptotically symmetric with respect to Ψ then*

$$2\mathscr{P}\left(\varphi, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right) \geq \mathscr{P}(2\varphi).$$

(2) *If φ is symmetric with respect to Ψ and $\mathscr{P}(2\varphi) < \infty$ then*

$$2\mathscr{P}\left(\varphi, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right) > \mathscr{P}(2\varphi) \text{ if and only if } 2\mathscr{P}(\varphi) \neq \mathscr{P}(2\varphi).$$

A sufficient condition for $2\mathscr{P}(\varphi) \neq \mathscr{P}(2\varphi)$ is that $\varphi \leq 0$, $\varphi^{-1}\{0\}$ is at most countable and $\inf\{s \in \mathbb{R} : \mathscr{P}(s\varphi) \leq 0\} = 2$. *Another sufficient condition is $\varphi \geq 0$, $\varphi^{-1}\{0\}$ is at most countable and* $\sup\{s \in \mathbb{R} : \mathscr{P}(s\varphi) \leq 0\} = 2$. In particular, $2\mathscr{P}(\varphi) \neq \mathscr{P}(2\varphi)$ holds if $\mathscr{P}(2\varphi) = 0$ and $\varphi < 0$.

Proof. Since Σ is finitely primitive and $(\Sigma \times G, \sigma \rtimes \Psi)$ is irreducible, one can easily verify that there exists a finite set $B \subset \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*$ such that for all $\omega_1, \omega_2 \in \Sigma^*$ there is $\tau(\omega_1, \omega_2) \in B$ with $\omega_1 \tau(\omega_1, \omega_2)\omega_2 \in \Sigma^*$. Hence, we can define a map $\Gamma : \Sigma^* \times \Sigma^* \to \Sigma^*$ given by $\Gamma(\omega_1, \omega_2) := \omega_1 \tau(\omega_1, \omega_2)\omega_2$, where $\tau(\omega_1, \omega_2) \in B$. Note that for each $n \in \mathbb{N}$, the restriction of Γ to $\Sigma^n \times \Sigma^*$ (resp. $\Sigma^* \times \Sigma^n$) is at most $\mathrm{card}(B)$-to-one. Setting $C_B := \min\left\{\inf S_{|\tau|}\varphi_{|[\tau]} : \tau \in B\right\} > -\infty$ and using the bounded distortion property of φ with constant $C_\varphi > 0$ (see Fact 2.2), we have for all $\omega_1, \omega_2 \in \Sigma^*$,

$$S_{\omega_1}\varphi + S_{\omega_2}\varphi - 2C_\varphi + C_B \leq \inf S_{|\omega_1|}\varphi_{|[\omega_1]} + \inf S_{|\omega_2|}\varphi_{|[\omega_2]} + \inf S_{|\tau(\omega_1,\omega_2)|}\varphi_{|[\tau(\omega_1,\omega_2)]} \leq S_{\Gamma(\omega_1,\omega_2)}\varphi.$$

Consequently, setting $l := \max\{|\tau| : \tau \in B\}$, we obtain for each sequence $(N_m) \in \mathbb{N}_0^{\mathbb{N}}$ and for each $n \in \mathbb{N}$,

$$
\begin{aligned}
&\operatorname{card}(B)\, \mathrm{e}^{2C_\varphi - C_B} \sum_{\omega \in \Sigma^* \cap \Psi^{-1}\{\mathrm{id}\} : 2n \le |\omega| \le 2n+N_n+l} \mathrm{e}^{S_\omega \varphi} \\
\ge\ & \sum_{g \in G} \left(\sum_{\omega_1 \in \Sigma^n \cap \Psi^{-1}\{g\}} \mathrm{e}^{S_{\omega_1}\varphi} \right) \left(\sum_{\omega_2 \in \Sigma^* \cap \Psi^{-1}\left\{g^{-1}\right\} : n \le |\omega_2| \le n+N_n} \mathrm{e}^{S_{\omega_2}\varphi} \right).
\end{aligned}
$$

Using that φ is asymptotically symmetric with respect to Ψ with sequences $(c_m) \in \mathbb{R}^{\mathbb{N}}$ and $(N_m) \in \mathbb{N}_0^{\mathbb{N}}$ as in Definition 4.3, it follows from the previous inequality that for all $n \in \mathbb{N}$,

$$
\begin{aligned}
&\operatorname{card}(B)\, \mathrm{e}^{2C_\varphi - C_B} \sum_{\omega \in \Sigma^* \cap \Psi^{-1}\{\mathrm{id}\} : 2n \le |\omega| \le 2n+N_n+l} \mathrm{e}^{S_\omega \varphi} \\
\ge\ & c_n^{-1} \sum_{g \in G} \left(\sum_{\omega_1 \in \Sigma^n \cap \Psi^{-1}\{g\}} \mathrm{e}^{S_{\omega_1}\varphi} \right) \left(\sum_{\omega_2 \in \Sigma^n \cap \Psi^{-1}\{g\}} \mathrm{e}^{S_{\omega_2}\varphi} \right) \qquad (4.1) \\
\ge\ & c_n^{-1} \sum_{g \in G} \left(\sum_{\omega_1 \in \Sigma^n \cap \Psi^{-1}\{g\}} \mathrm{e}^{2S_{\omega_1}\varphi} \right) = c_n^{-1} \sum_{\tau \in \Sigma^n} \mathrm{e}^{2S_\tau \varphi}.
\end{aligned}
$$

Using that $\lim_n c_n^{1/n} = 1$ it follows from (4.1) that

$$
\limsup_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in \Sigma^* \cap \Psi^{-1}\{\mathrm{id}\} : 2n \le |\omega| \le 2n+N_n+l} \mathrm{e}^{S_\omega \varphi} \ge \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\tau \in \Sigma^n} \mathrm{e}^{2S_\tau \varphi} = \mathscr{P}(2\varphi).
$$

Finally, using that $\lim_n n^{-1}(N_n + l) = 0$, one verifies that

$$
2\mathscr{P}\left(\varphi, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right) \ge \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in \Sigma^* \cap \Psi^{-1}\{\mathrm{id}\} : 2n \le |\omega| \le 2n+N_n+l} \mathrm{e}^{S_\omega \varphi},
$$

which finishes the proof of the first assertion.

We now turn to the proof of the second assertion. First note that by passing to the potential $\varphi - \mathscr{P}(2\varphi)/2$ we may without loss of generality assume that $\mathscr{P}(2\varphi) = 0$. By item (1) it remains to show that $\mathscr{P}\left(\varphi, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right) = 0$ if and only if $\mathscr{P}(\varphi) = 0$. Since $\mathscr{P}\left(\varphi, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right) \ge \mathscr{P}(2\varphi)/2 = 0$ by item (1), we deduce that $\mathscr{P}\left(\varphi, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right) = 0$ whenever $\mathscr{P}(\varphi) = 0$. Now, for the opposite implication, suppose that $\mathscr{P}\left(\varphi, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right) = 0$. Since Σ is finitely primitive and $\mathscr{P}(2\varphi) = 0$ there exists a unique σ-invariant Gibbs measure for 2φ by Theorem 2.6, such that for each $n \in \mathbb{N}$,

$$
C_{2\varphi} \sum_{\tau \in \Sigma^n} \mathrm{e}^{2S_\tau \varphi} \ge \sum_{\tau \in \Sigma^n} \mu_{2\varphi}[\tau] = 1.
$$

Since φ is symmetric with respect to Ψ and by (4.1) there exists $C > 0$ such that for all $n \in \mathbb{N}$,

$$
\sum_{\omega \in \Sigma^* \cap \Psi^{-1}\{\mathrm{id}\} : 2n \le |\omega| \le 2n+N_n+l} \mathrm{e}^{S_\omega \varphi} \ge C \sum_{\tau \in \Sigma^n} \mathrm{e}^{2S_\tau \varphi} \ge CC_{2\varphi}^{-1} > 0. \qquad (4.2)
$$

Next, by choosing a sequence $(n_k) \in \mathbb{N}^{\mathbb{N}}$ tending to infinity, such that $2n_{k+1} > 2n_k + N_{n_k} + l$ for all $k \in \mathbb{N}$, we deduce from (4.2) that

$$
\sum_{n \in \mathbb{N}} \sum_{\omega \in \Sigma^n \cap \Psi^{-1}\{\mathrm{id}\}} \mathrm{e}^{S_\omega \varphi} = \infty. \qquad (4.3)
$$

Since $\mathscr{P}\left(\varphi, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right) = 0$, (4.3) implies that $\varphi \circ \pi_1$ is a recurrent with respect to $\sigma \rtimes \Psi$. By Theorem 4.6 we then have that $0 = \mathscr{P}\left(\varphi, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right) = \mathscr{P}(\varphi)$.

Finally, we show that a sufficient condition for $2\mathscr{P}(\varphi) \neq \mathscr{P}(2\varphi)$ is given by $\varphi \leq 0$, $\varphi^{-1}\{0\}$ at most countable and $\inf\{s \in \mathbb{R} : \mathscr{P}(s\varphi) \leq 0\} = 2$. Suppose for a contradiction that $2\mathscr{P}(\varphi) = \mathscr{P}(2\varphi)$. Similarly, as in the proof of Proposition 3.12, one verifies that $\mathscr{P}(2\varphi) \leq 0$. We now consider two cases.

Case 1: Suppose that $\mathscr{P}(2\varphi) = 0$. By convexity of topological pressure we conclude that the map $s \mapsto \mathscr{P}(s\varphi)$, $s \in [1,2]$, is a real-valued, convex and continuous function satisfying $\mathscr{P}(\varphi) = \mathscr{P}(2\varphi) = 0$. It is well-known that $s \mapsto \mathscr{P}(s\varphi)$ is real analytic on $(1,2)$ and that for each $s_0 \in (1,2)$, $\frac{\partial}{\partial s}\mathscr{P}(s\varphi)\big|_{s=s_0} = \int \varphi \, d\mu_{s_0\varphi}$, where $\mu_{s_0\varphi}$ denotes the unique σ-invariant Gibbs measure for $s_0\varphi$ (see [RSU08]). Since $\mu_{s_0\varphi}$ has no atoms, $\varphi^{-1}\{0\}$ is at most countable and $\varphi \leq 0$, it follows that $\int \varphi \, d\mu_{s_0\varphi} < 0$ for each $s_0 \in (1,2)$, which gives the desired contradiction.

Case 2: Suppose that $\mathscr{P}(2\varphi) < 0$. Since $2\mathscr{P}(\varphi) = \mathscr{P}(2\varphi)$, we have $\mathscr{P}(\varphi) < 0$, contradicting that $\inf\{s \in \mathbb{R} : \mathscr{P}(s\varphi) \leq 0\} = 2$.

The proof of sufficiency of $\varphi \geq 0$, $\varphi^{-1}\{0\}$ at most countable and $\sup\{s \in \mathbb{R} : \mathscr{P}(s\varphi) \leq 0\} = 2$ can be proved in an analogue fashion. The proof is complete. □

4.2.1. *An application to cogrowth of group presentations.* At the end of this section, we would like to apply Proposition 4.7 to give a new proof of an estimate on the cogrowth of group presentations. Let $\langle g_1, \dots, g_d | r_1, r_2, \dots \rangle$, $d \geq 2$, be the presentation of a finitely generated group G which is not free. Then, $G = \mathbb{F}_d / N$ where $\mathbb{F}_d = \langle g_1, \dots, g_d \rangle$ denotes the free group and N denotes the normal subgroup $\mathbb{F}_d$ generated by $r_1, r_2, \dots$. Since G is not free, we have that N is non-trivial. The cogrowth γ of $\langle g_1, \dots, g_d | r_1, r_2, \dots \rangle$, which was independently introduced by Grigorchuk ([Gri80]) and Cohen ([Coh82]), is given by

$$\gamma := \limsup_{n \to \infty} \left(\operatorname{card}\left(\{w \in \mathbb{F}_d \cap N : |w| = n\}\right)\right)^{1/n},$$

where $|w|$ refers to the word length of $w \in \mathbb{F}_d$. We clearly have that $\gamma \leq 2d - 1$. Further, it is easy to show that $\gamma \geq (2d-1)^{1/2}$. The following stronger result was proved in [Coh82], using a generating function technique ([Coh82, Theorem 3]) and an estimate of Kesten ([Kes59b, Theorem 3]). We obtain this result as a corollary of Proposition 4.7.

Corollary 4.8. *Let $\langle g_1, \dots, g_d | r_1, r_2, \dots \rangle$, $d \geq 2$, be the presentation of a finitely generated group G which is not free. Then, for the cogrowth γ of $\langle g_1, \dots, g_d | r_1, r_2, \dots \rangle$ we have $\gamma > (2d-1)^{1/2}$.*

Proof. For $I := \left\{g_1^{\pm 1}, \dots, g_d^{\pm 1}\right\}$ consider the Markov shift $\Sigma := \left\{\omega \in I^{\mathbb{N}} : \forall i \in \mathbb{N}\ \omega_i \neq (\omega_{i+1})^{-1}\right\}$. For the non-trivial normal subgroup N of $\mathbb{F}_d$ which is generated by $r_1, r_2, \dots$, let $\Psi_N : I^* \to G$ denote the unique semigroup homomorphism such that $\Psi_N(g) = g \bmod N$, for each $g \in I$.

We can now apply Proposition 4.7 (2) to the constant zero potential $0 : \Sigma \to \{0\}$, which is symmetric with respect to Ψ_N. Since $\mathscr{P}(0) = \log(2d-1) \neq 0$, the proposition implies that

$$2\mathscr{P}\left(0, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right) > \log(2d-1).$$

Observing that $\gamma = \mathrm{e}^{\mathscr{P}\left(0, \Psi^{-1}\{\mathrm{id}\} \cap \Sigma^*\right)}$, we obtain that $\gamma > (2d-1)^{1/2}$. The proof is complete. □

5. PROOF OF THE MAIN RESULTS

For a GDMS Φ associated to $\mathbb{F}_d = \langle g_1, \dots, g_d \rangle$, $d \geq 2$, we set $I := \left\{ g_1^{\pm 1}, \dots, g_d^{\pm 1} \right\}$ and we introduce the Markov shift

$$\Sigma := \left\{ \omega \in I^{\mathbb{N}} : \forall i \in \mathbb{N}\ \omega_i \neq (\omega_{i+1})^{-1} \right\}.$$

One immediately verifies that Σ is finitely primitive.

For a non-trivial normal subgroup N of $\mathbb{F}_d$, let $G := \mathbb{F}_d/N$ and let $\Psi_N : I^* \to G$ denote the unique semigroup homomorphism such that $\Psi_N(g) = g \bmod N$, for each $g \in I$. Using that $d \geq 2$ and that N is a non-trivial normal subgroup of $\mathbb{F}_d$, one immediately verifies that the group-extended Markov system $\sigma \rtimes \Psi_N : \Sigma \times G \to \Sigma \times G$ is irreducible.

Define the potential $\varphi : \Sigma \to \mathbb{R}^-$ given by

$$\varphi(\omega) := \zeta_\Phi((\omega_1, \omega_2), (\omega_2, \omega_3), \dots).$$

By Fact 3.5 we have that ζ_Φ is Hölder continuous, which implies that also φ is Hölder continuous. Since the Lipschitz constants of the conformal contractions of Φ are bounded away from one, we have that φ is bounded away from zero. Hence, by Fact 2.4 applied to $\mathscr{P}_{-\varphi}(0, \Sigma^*)$ and $\mathscr{P}_{-\varphi}\left(0, \Psi_N^{-1}\{\mathrm{id}\} \cap \Sigma^*\right)$ respectively, we conclude that

$$\delta(\mathbb{F}_d, \Phi) \text{ is the unique } s \in \mathbb{R} \text{ such that } \mathscr{P}(s\varphi) = 0 \tag{5.1}$$

and

$$\delta(N, \Phi) \text{ is the unique } s \in \mathbb{R} \text{ such that } \mathscr{P}\left(s\varphi, \Psi_N^{-1}\{\mathrm{id}\} \cap \Sigma^*\right) = 0. \tag{5.2}$$

We also observe that the maps $s \mapsto \mathscr{P}(s\varphi)$ and $s \mapsto \mathscr{P}\left(s\varphi, \Psi_N^{-1}\{\mathrm{id}\} \cap \Sigma^*\right)$ are strictly decreasing, real-valued functions on $\mathbb{R}$.

Finally, let us clarify how the symmetry assumptions imposed on Φ are reflected in properties of φ. Clearly, if Φ is symmetric then φ is symmetric with respect to Ψ_N, and if Φ is weakly symmetric then φ is asymptotically symmetric with respect to Ψ_N.

Proof of Theorem 1.1 . *Ad (1):* By (5.1) and (5.2), our assumption $\delta(\mathbb{F}_d, \Phi) = \delta(N, \Phi)$ implies that

$$\mathscr{P}\left(\delta(\mathbb{F}_d, \Phi)\varphi, \Psi_N^{-1}\{\mathrm{id}\} \cap \Sigma^*\right) = \mathscr{P}(\delta(\mathbb{F}_d, \Phi)\varphi) = 0.$$

Applying Theorem 4.5 to the Hölder continuous potential $\delta(\mathbb{F}_d, \Phi)\varphi$ shows that $G = \mathbb{F}_d/N$ is amenable.

Ad (2): By (5.1), we have that $\mathscr{P}(\delta(\mathbb{F}_d, \Phi)\varphi) = 0$. Since the GDMS Φ associated to $\mathbb{F}_d$ is weakly symmetric, we have that $\delta(\mathbb{F}_d, \Phi)\varphi$ is asymptotically symmetric with respect to Ψ_N. Since G is amenable, applying Theorem 4.4 to the Hölder continuous potential $\delta(\mathbb{F}_d, \Phi)\varphi$, implies that $\mathscr{P}\left(\delta(\mathbb{F}_d, \Phi)\varphi, \Psi_N^{-1}\{\mathrm{id}\} \cap \Sigma^*\right) = \mathscr{P}(\delta(\mathbb{F}_d, \Phi)\varphi) = 0$. Consequently, by (5.2), we have $\delta(\mathbb{F}_d, \Phi) = \delta(N, \Phi)$.

Ad (3): For the first assertion let $\varepsilon > 0$. We apply Proposition 4.7 (1) to the Hölder continuous potential $(\delta(\mathbb{F}_d, \Phi) - \varepsilon)\varphi/2$, which is asymptotically symmetric with respect to Ψ_N. We have

$$2\mathscr{P}\left((\delta(\mathbb{F}_d, \Phi) - \varepsilon)\varphi/2, \Psi_N^{-1}\{\mathrm{id}\} \cap \Sigma^*\right) \geq \mathscr{P}((\delta(\mathbb{F}_d, \Phi) - \varepsilon)\varphi). \tag{5.3}$$

Since $\mathscr{P}((\delta(\mathbb{F}_d,\Phi)-\varepsilon)\varphi)>0$ by (5.1), we have $\mathscr{P}\left((\delta(\mathbb{F}_d,\Phi)-\varepsilon)\varphi/2,\Psi_N^{-1}\{\mathrm{id}\}\cap\Sigma^*\right)>0$ by (5.3), which then implies that $\delta(N,\Phi)>(\delta(\mathbb{F}_d,\Phi)-\varepsilon)/2$ by (5.2). Since ε was chosen to be arbitrary, the result follows.

We now turn to the second assertion. We can apply Proposition 4.7 (2) to the Hölder continuous potential $\delta(\mathbb{F}_d,\Phi)\varphi/2$, which is symmetric with respect to Ψ_N. Since $\mathscr{P}(\delta(\mathbb{F}_d,\Phi)\varphi)=0$ and $\delta(\mathbb{F}_d,\Phi)\varphi<0$, we obtain that $\mathscr{P}\left(\delta(\mathbb{F}_d,\Phi)\varphi/2,\Psi_N^{-1}\{\mathrm{id}\}\cap\Sigma^*\right)>0$. Hence, $\delta(N,\Phi)>\delta(\mathbb{F}_d,\Phi)\varphi/2$ by (5.2).

Ad (4): Since $P(\delta(N,\Phi),N,\Phi)=\infty$ and $\mathscr{P}\left(\delta(N,\Phi)\varphi,\Psi_N^{-1}\{\mathrm{id}\}\cap\Sigma^*\right)=0$ by (5.2), we have that $\delta(N,\Phi)\varphi\circ\pi_1:\Sigma\times G\to\mathbb{R}^-$ is recurrent (with respect to $\sigma\rtimes\Psi_N$). Since the Hölder continuous potential $\delta(N,\Phi)\varphi\circ\pi_1$ is asymptotically symmetric with respect to Ψ_N, Theorem 4.6 implies that $\mathscr{P}(\delta(N,\Phi)\varphi)=\mathscr{P}\left(\delta(N,\Phi)\varphi,\Psi_N^{-1}\{\mathrm{id}\}\cap\Sigma^*\right)$. Since the latter pressure is equal to zero, we obtain that $\delta(\mathbb{F}_d,\Phi)=\delta(N,\Phi)$ by (5.1). □

Proof of Corollary 1.2 . First suppose that N is finitely generated. Then, the edge set $\tilde{E}$ of the N-induced GDMS $\tilde{\Phi}$ is finite (see Definition 3.9). Combining with the fact that $\zeta_{\tilde{\Phi}}:\Sigma_{\tilde{\Phi}}\to\mathbb{R}^-$ is bounded away from zero, it follows from Fact 2.4 that $\mathscr{P}_{-\zeta_{\tilde{\Phi}}}\left(0,\Sigma^*_{\tilde{\Phi}}\right)$ is the unique zero of the map $s\mapsto\mathscr{P}\left(s\zeta_{\tilde{\Phi}},\Sigma^*_{\tilde{\Phi}}\right)$. Consequently, by Proposition 3.12, we have that $\mathscr{P}\left(\delta(N,\Phi)\zeta_{\tilde{\Phi}},\Sigma^*_{\tilde{\Phi}}\right)=0$ and $P(\delta(N,\Phi),N,\Phi)=\infty$. Hence, Theorem 1.1 (4) implies that $\delta(N,\Phi)=\delta(\mathbb{F}_d,\Phi)$, which then gives that $\mathbb{F}_d/N$ is amenable by Theorem 1.1 (1).

Now suppose that $\mathbb{F}_d/N$ is non-amenable. By the first assertion of the corollary, we have that N is infinitely generated. By Theorem 1.1 (1) we have that $\delta(N,\Phi)<\delta(\mathbb{F}_d,\Phi)$, which then implies that $P(N,\delta(N,\Phi),\Phi)<\infty$ by Theorem 1.1 (4). □

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